MATH 1650: SECTION 9.4: THE BINOMIAL THEOREM

Simply stated, the Binomial Theorem is a formula for the expansion of quantities $(a+b)^n$ for natural numbers n. In High School Algebra, you probably have seen specific instances of the formula, namely

$$\begin{aligned}(a+b)^1 &= a+b\\(a+b)^2 &= a^2+2ab+b^2\\(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

If we wanted the expansion for $(a+b)^4$ we would write $(a+b)^4=(a+b)(a+b)^3$ and use the formula that we have for $(a+b)^3$ to get $(a+b)^4=(a+b)\left(a^3+3a^2b+3ab^2+b^3\right)=a^4+4a^3b+6a^2b^2+4ab^3+b^4$. We wish to find a generic formula for $(a+b)^n$ we can apply directly without working through formulas for lesser powers than n. A key player in what is to come is the factorials (whom we encountered in section 9.1)

Recall the number $n!$ (read 'n-factorial') is defined as follows: $0!=1$ and $n!=n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1$.

EXAMPLE: Simplify the following expressions.

1. $\dfrac{3!\,2!}{0!}$

2. $\dfrac{7!}{5!}$

3. $\dfrac{1000!}{998!\,2!}$

4. $\dfrac{(k+2)!}{(k-1)!}$, $k\geq 1$

BINOMIAL COEFFICIENTS: Given two whole numbers n and j with $n \geq j$, the binomial coefficient $\binom{n}{j}$ (read, 'n choose j') is the whole number given by

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

For now, we can physically interpret $\binom{n}{j}$as the number of ways to select j items from n items where the order of the items selected is unimportant. For example, suppose you won two free tickets to a special screening of the latest Hollywood blockbuster and have five good friends each of whom would love to accompany you to the movies. There are $\binom{5}{2}$ ways to choose who goes with you. Working this out, we find:

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{5 \cdot 4}{2} = 10$$

So there are 10 different ways to distribute those two tickets among five friends.

EXAMPLE: To play 'Mega Millions,' a person selects 5 distinct numbers from 1-70 (order unimportant) along with a 'power ball' number 1-25.

5. Use a binomial coefficient to determine how many ways there are to choose 5 numbers from 1 - 70.

6. How many different tickets are possible?

   HINT: For each choice of numbers 1-70, there are 25 tickets possible, depending on the choice of the 'power ball' number ...

THE BINOMIAL THEOREM: For nonzero real numbers a and b and for all natural numbers n.

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j$$

To get a feel of what this theorem is saying and how it really isn't as hard to remember as it may first appear, let's consider the specific case of $n = 4$. According to the theorem, we have

$$\begin{aligned}
(a+b)^4 &= \sum_{j=0}^{4} \binom{4}{j} a^{4-j} b^j \\
&= \binom{4}{0} a^{4-0} b^0 + \binom{4}{1} a^{4-1} b^1 + \binom{4}{2} a^{4-2} b^2 + \binom{4}{3} a^{4-3} b^3 + \binom{4}{4} a^{4-4} b^4 \\
&= \binom{4}{0} a^4 + \binom{4}{1} a^3 b + \binom{4}{2} a^2 b^2 + \binom{4}{3} ab^3 + \binom{4}{4} b^4 \\
&= a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4
\end{aligned}$$

DISCUSS: What patterns do you see on... the exponents? the coefficients?

EXAMPLE: Use the Binomial Theorem to find the following.

7. $(x-2)^4$

8. 2.1^3

   HINT: $2.1^3 = (2+0.1)^3 \ \ldots$

9. The term containing x^3 in the expansion $(2x+y)^5$

THE BINOMIAL DISTRIBUTION:

Suppose an experiment has a probability p of 'success' and a probability of $1-p$ of 'failure.' For instance, suppose we roll a 'fair' six-sided die. Let us say a 'success' is rolling a four. Then the probability here of a success is $p = \frac{1}{6}$ while the probability of failure here, or *not* rolling a four, is $1 - \frac{1}{6} = \frac{5}{6}$.

If we run this experiment n times, then the probability of *exactly* j successes is given by $\dbinom{n}{j} p^j (1-p)^{n-j}$.

Here, the binomial coefficient counts the number of ways we can produce j successes out of n trials. The 'bi' in 'binomial' comes from the fact that each trial produces one of two outcomes: a 'success' (with a probability of p) or 'failure' (with probability $1-p$). If we roll the fair die 5 times, the probability we get *exactly* 2 fours is:

$$\binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{5-2} = \frac{625}{3888} \approx 16\%$$

EXAMPLE: Suppose we roll a fair die 5 times.

10. What is the probability of rolling *at least* 2 fours on 5 rolls?

PASCAL'S TRIANGLE:

Pascal's Triangle is named in honor of mathematician Blaise Pascal. Pascal's Triangle gives us a quick method to generate all the binomial coefficients for a given binomial expansion. Below we attempt to demonstrate this building process to generate the first five rows of Pascal's Triangle.

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ & & \searrow\swarrow & & \\ \boxed{1} & & \underline{1+1} & & \boxed{1} \end{array} \longrightarrow \begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ & & \searrow\swarrow & & \searrow\swarrow & & \\ \boxed{1} & & \underline{1+2} & & \underline{2+1} & & \boxed{1} \end{array} \longrightarrow \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ & & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \\ \boxed{1} & & \underline{1+3} & & \underline{3+3} & & \underline{3+1} & & \boxed{1} \end{array} \longrightarrow \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

To see how we can use Pascal's Triangle to expedite the Binomial Theorem, suppose we wish to expand $(3x-y)^4$. The coefficients we need are $\binom{4}{j}$ for $j = 0, 1, 2, 3, 4$ and are the numbers which form the fifth row of Pascal's Triangle. (The row with the '4' as the second entry.)

Since we know that the exponent of $(3x)$ in the first term is 4 and then decreases by one as we go from left to right while the exponent of $(-y)$ starts at 0 in the first term and then increases by one as we move from left to right, we quickly obtain

$$\begin{aligned} (3x-y)^4 &= (1)(3x)^4 + (4)(3x)^3(-y) + (6)(3x)^2(-y)^2 + 4(3x)(-y)^3 + 1(-y)^4 \\ &= 81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4 \end{aligned}$$

EXAMPLE:

1. Create the sixth row of Pascal's Triangle

2. Use Pascal's Triangle to expand $(x + 2y)^5$

HOMEWORK: Section 9.4: 1 - 25 odd.